Petri Nets:
an introduction

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Agenda

- Definitions and Modeling
- Dynamic analysis
- Static analysis
Definition and Modeling
Quick History

- Petri Nets were invented in the 60’s by Carl Adam Petri

- In the same period other models were introduced:
  - Transition Systems
  - Vector Addition System (VAS)

- Other formal models were introduced later:
  - Milner’s Calculus for Communicating Systems (CCS)
  - Ho’s Discrete Events Dynamic Systems (DEDS)

- Motivation was to model the notion of asynchronous parallelism or concurrency
  - A finite State Machine (FSM) behaves fundamentally sequentially
Models: quick comparisons

- In term of power of description:
  FSM < PN < Turing Machines
- Despite defined independently, it was proven that PN and VAS were ‘somewhat’ equivalent
- Parallel programs are seen as a set of sequential processes sharing resources (including exchanging messages)
  - Each sequential process can be seen as a Finite State Machine
A transition system is a tuple \((S, T, \rightarrow)\) where:
- \(S\) is a set of states,
- \(T\) is a finite set of transitions and
- \(\rightarrow \subseteq S \times T \times S\) relates transitions with states.

Each transition can be seen as a function from \(S\) to \(S\).

Most of the time, an element \(s_0\) of \(S\) is distinguished and called initial state.

Most of the time, \(S\) will be an element of \(\mathbb{N}^d\) to model amounts of \(d\) different kind of resources the system disposes of.

A trajectory (a run, or a trace) is a:
- word of \(T^*\)
- sequence \(s_1, s_2, s_3, \ldots s_k\) such that \(\forall\ i < k, \exists\ t \in T\), such that \(s_i \rightarrow t s_{i+1}\)
Vector Addition Systems (ref: J. Leroux)

Definition

A **vector addition system (VAS)** is a finite set $A \subseteq \mathbb{Z}^d$.

* $A$ set of actions.
* $\mathbb{N}^d$ set of markings.

A **run** is a non-empty word $\rho = m_0 \ldots m_k$ of markings such that:

$$\forall j \in \{1, \ldots, k\} \quad m_j \in m_{j-1} + A$$

In this case, $m_k$ is said to be **reachable** from $m_0$.

Theorem (Mayr 1981, Kosaraju 1982)

*The reachability problem is decidable.*
Reachability

Example

\[ A = \{ \text{arrows} \} \]

\[ \rho = (0, 2) (1, 3) (2, 4) (3, 5) (4, 6) (3, 4) (2, 2) (1, 0) \]

\( n \) is reachable from \( m \).
Unreachability

Example

\[ A = \{ a, b \} \]

\[ \phi(x_1, x_2) := 0 \leq x_1 \land 0 \leq x_2 \land x_2 \leq x_1 + 2 \]

\text{Reachable markings}

\text{\textbf{n} is not reachable from \textbf{m}.}
A Petri Net is a class of transition systems where P is a set of places (circles), T a set of transitions (rectangles), an initial marking (or initial state) Mo is a function from P to \(\mathbb{N}\).

- We dynamically move from one state to the next by ‘firing’ (executing, enabling) a transition \(t\) from \(M\) iff its preconditions (Pre) hold
  - Pre: \(P \times T \rightarrow \mathbb{N}\) gives the weight from places to a transitions
  - \(M \geq \text{Pre}(.,t)\)
- The relationship between P and T is described by a labeled bigraph \(\langle P, T; \text{Pre}, \text{Post}\rangle\) also represented by an incidence matrix \(C = \text{Post} - \text{Pre}\)
  - Post: \(P \times T \rightarrow \mathbb{N}\) gives the weight from transitions to places
- A sequence of transitions (or trace) \(R\) is associated with a vector \(\vec{R}\) such that \(R_i\) is the number of occurrences of the transition \(t_i\) in \(R\)
- From Mo, we reach M by enabling R with the equation:
  - \(M = CR + Mo\)
Example: Vending Machine (Token Games)

- Deposit 5c
- Deposit 10c
- Take 15c bar
- Take 20c bar

States:
- '0c'
- '5c'
- '10c'
- '15c'
- '20c'
Test to 0 and Priority

- **Inhibitor edge:**
  t is fireable iff $M(p) = 0$

- **Priority:**
  An order is defined over $T$, t is fireable from $M$ iff
  
  $M \geq \text{Pre}(.,t)$
  
  $\exists \ t' > t, \ M \geq \text{Pre}(.,t')$

We can model an inhibitor edge between p and t using priorities

By adding $t'$ such that: $t' > t$ and adding a loop between $p$ and $t'$: t can't fire as long as there is a token in $p$
Dynamic analysis
General properties

- A marking $M$ is **reachable** iff it exists a sequence going from $M_0$ to $M$
- A transition $t$ is live iff from any reachable marking $M$ it exists a sequence reaching a marking $M'$ allowing to fire $t$
- A Petri Net is **live** iff all transition is live

- A place is bounded by an integer $k>0$ iff it does not exist a reachable marking $M$ such that $M(p)<k$
- A Petri Net is **bounded** iff all place is bounded

- A marking $H$ is a **home state** iff from any reachable marking $M$, it is possible to reach $H$
Modeling with a Petri Net
Linear invariant linked to \{okSel, Hang-up\}
Reachability Graph

Idle

W1 Idle pick-up

W1 ok-Sel Conv

W1, ok-Sel, free Pkd

Wf1 free Idle

Wf2 free Idle

Whg Idle

Wf2 Conv hang-up

Wf1 Conv hang-up

Wf1, free Pkd hang-up

Wf2, free Pkd hang-up

Whg Pkd hang-up

Conv

Sel ok-Sel Conv

Sel ok-Sel, free Pkd

Conv, free Pkd

Pkd Pkd

Idle

Pkd, hang-up

Pick-up
A behavior that does not go back 'home'
Reduction
Place and transition aggregation
Identifying identical subgraphs

Diagram showing states and transitions:
- Idle
- W1-Sel
- Conv
- OkSel
- Free
- Hang up
- Pick up
- Conv
- Pkd

Transitions labeled with numbers:
1. Wf1 to Idle
2. W1-Sel to OkSel
3. Conv to Free
4. Pkd to Conv
5. Conv to Wf1
6. Pkd to Wf1
7. Free to Conv
8. Conv to Pkd
9. Pkd to Conv
10. Conv to Wf1
11. Wf1 to Pkd
12. Pkd to Wf1
13. Wf1 to W1-Sel
14. Conv to Pkd
15. Pkd to Conv
16. Conv to Pkd
17. Pkd to Conv
Reducing identical subgraph

Diagram showing states and transitions:
- LA
- idle
- Sel
- D
- 1
- Pick up
- okSel
- R
- 14
- Conv
- 15
- pkd
- D
- 17
- Hang up
- free
- 16
- Wf1
- 9
- (6,12-13)
- Conv
- 5
- 3
- 4
- pkd
- 11
- 12
- 13
- 14
Simplified Reachability Graph
Coverability Tree
Coverability Tree

\[ R = \langle P, T; \text{Pre}, \text{Post}; \text{Mo} \rangle; \ AC(R) = \langle S, X \rangle \text{ where } S \in \mathbb{N}^d \text{, where } d = |P|, \]

- Mo labels \( r \) the root of the coverability tree \( AC \)
- \( s \) of \( S \) labeled by \( Q \) of \( \mathbb{N}^\omega \) has no successor iff
  - Either \( \exists \ s' \) of \( S \) also labeled by \( Q \) in the path from \( r \) to \( s \)
  - Or there is no transition fireable from \( Q \)
- If not, there are transitions fireable from \( Q \). Then for each such \( t \), we build a new edge labeled by \( t \) and ending by a node \( s' \) labeled by \( Q' \) such that for each place \( i \) of \( P \):
  - \( Q'i = \omega \) iff \( \exists \) on the path from \( r \) to \( s' \) a label \( Q'' \) such that:
    - \( Q'' \leq Q + \text{Post}(.,t) - \text{Pre}(.,t) \) and \( Q''i < Qi + \text{Post}(i,t) - \text{Pre}(i,t) \)
  - \( Q'i = Qi + \text{Post}(i,t) - \text{Pre}(i,t) \) otherwise

Theorem (Karp & Miller 1969)

\( AC(R) \) is finite for any Petri Net \( R \)
Coverability Tree: an example (1/2) (ref: Hermann & Lin)

Initialization: \[ M_0 = (1,0,0) \text{ new} \]

Step 1:
\[ t_1 \xrightarrow{(1,0,0) \text{ new}} t_2 \]
\[ m_1 = (1,1,0) \geq (1,0,0) \]
\[ \text{new } m_1 = (1,\omega,0) \quad m_2 = (0,1,1) \text{ new} \]

Step 2 (\(m_1\)):
\[ t_1 \xrightarrow{(1,0,0) \text{ new}} t_2 \]
\[ \text{new } m_1 = (1,\omega,0) \quad m_2 = (0,1,1) \text{ new} \]
\[ \text{old } (1,\omega,0) \quad \text{new } m_3 = (0,\omega,1) \]
Coverability Tree: an example (2/2)

Step 3 (m3):

Nothing is fireable from m4
Limitations

- State Space explosion
- Unbounded resources are identified, however, the coverability tree can hardly be used to prove any property, for instance: “Can a given state be reached?”
  - This question has been proven decidable for Petri Nets or equivalent models (see slide Vector Addition Systems (ref: J. Leroux))
- Parameterized models are more difficult to take into account
Static analysis
A linear invariant is associated with a semiflow \( f \) of \( \mathbb{N} \left| P \right| \) such that

- \( f^T C = 0 \) (the Kichhoff law is verified for each transition)
- the scalar product \( f^T M \) is then invariant for any reachable marking:
  \[ f^T M = \sum_{p \in P} f(p) M(p) \]

If \( f^T M > 2 \) and is odd then the net is live; if \( f^T M \) is even then the net is not live.

Liveness is a non monotonic property!

Adding more token does not necessarily bring liveness.
General Bibliography on Petri Nets and other models

- GW Brams “Réseaux de Petri, Théorie et Pratique” tome 1, Masson, 1983
- The World of Petri nets: [http://www.informatik.uni-hamburg.de/TGI/PetriNets/](http://www.informatik.uni-hamburg.de/TGI/PetriNets/)
References

- Slides 7, 8, 9 are from Leroux J. “Vector Addition System Reachability Problem” Labri France, 2011,

- Slides 24, 25 are from Hermann J. & Lin E. “Petri Nets: Tutorial and Applications” Nov. 1997,